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A Bernasconi Model for Constructing Ground-State Spin Systems

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Abstract—A Bernasconi model for a one-dimensional chain of quantum particles is considered. It is shown that searching for the ground state of such a quantum system is equivalent to searching for binary sequences with minimum energy levels of the side lobes of the aperiodic autocorrelation function (ACF). A review and new results regarding the construction of such binary sequences are presented.

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INTRODUCTION

Classical lattice systems allowing us to describe interatomic potential using finite numbers of values in points defined by the lattice nodes are widely used in statistical physics. The simplest of these is the one-dimensional Ising model [1]. Within such a model, the spin can assume one of two values in each lattice node with number n: $s_n = \pm 1$. In this work, we consider a one-dimensional spin lattice $\vec{S} = (s_0, s_1, \ldots, s_{N-1})$ consisting of N particles with equal distances between neighboring particles.

THEORETICAL

The energy of such a quantum system consists of pairwise exchange interaction between neighboring atom spins and the interaction of the spins and an external magnetic field:

$$E(S) = -\sum_{n,\tau} J_{n\tau} s_n s_\tau - \sum_n h_n s_n, \qquad (1)$$

where indices n and τ denote the number of lattice nodes, h_n is the strength of the external magnetic field in the nth node, and $J_{n\tau}$ is the energy of interaction for spins located in nodes n and τ.

With external magnetic field $h = 0$, any energy level is doubly degenerate, since the energy of interaction does not change upon the rotation of all spins. In the simplest case, the energy of interaction can be considered identical for all pairs of neighboring atoms; i.e., $J_{n\tau} = J$. As a result, we obtain the following model for the quantum system's energy:

$$E(S) = -J \sum_{n,\tau} s_n s_\tau. \qquad (2)$$

In [2], Bernasconi pointed out the connection between the problem of searching for low-energy states of a simplified one-dimensional Ising model (2) and that of constructing binary sequences $\vec{S} = (s_0, s_1, \ldots, s_{N-1})$ of N length with low levels of autocorrelation.

Aperiodic autocorrelation function $\vec{C} = (c_0, c_1, \ldots, c_{N-1})$ of the binary sequence can be written in the form

$$c_\tau = \sum_{n=0}^{N-1-\tau} s_n \cdot s_{n+\tau}, \qquad (3)$$

where $\tau = 0, 1, \ldots, N-1$.

In Bernasconi's model, the Hamiltonian of the quantum system of the simplest one-dimensional Ising model is equal to the energy of the side lobes of the aperiodic autocorrelation function:

$$\mathrm{H}(\vec{S}) = \sum_{\tau=1}^{N-1} C_\tau^2. \qquad (4)$$

The problem of constructing the simplest quantum system in the form of a one-dimensional Ising model is thus reduced to one of constructing binary sequences with the lowest energy of the side lobes.

There are two criteria for the optimality of binary sequences with low levels of aperiodic autocorrelation. The first is the minimax criterion, in which the maximum level of side lobe PSL

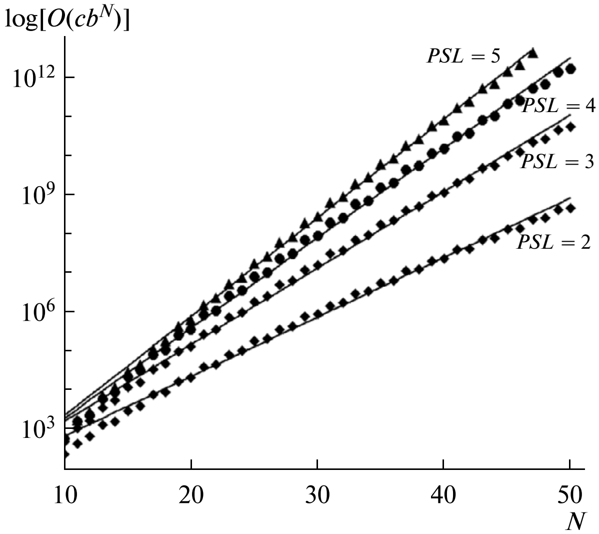
$$PSL(C) = \max_{1 \le \tau \le N-1} |C_\tau| \qquad (5)$$

must be minimal:

$$MPS = \min_S PSL. \qquad (6)$$

The second is the MF (merit factor) coefficient characterizing the ratio of the main frame energy to that of the side lobes of the aperiodic aurocorrelation function:

$$MF(C) = \frac{N^2}{2\sum_{\tau=1}^{N-1} [C_\tau]^2}. \qquad (7)$$



Computational complexity of the new algorithm for global searches for optimum minimax sequences.

The optimum binary sequence of a given length N has the highest value of the MF coefficient.

Binary sequences that are optimal according to these two criteria are generally different, but for an entire set of lengths they are identical. Below, we present new results in the area of constructing minimax optimal binary sequences.

A NEW GLOBAL SEARCH ALGORITHM

There are two groups of algorithms used to search for optimal binary sequences. Local algorithms have an efficiency approximately equal to $O(1.4^N)$, while that of global ones is approximately $O(1.85^N)$. Despite the lower performance, only global algorithms ensure the success of an optimum sequence search.

The following results are well-known in the field of constructing optimum minimax sequences with the use of global search algorithms. In [3], an exhaustive search for minimax binary sequences was performed for lengths $N \leq 40$. In [4], this list was extended to lengths $N \leq 48$. In [5], an exhaustive search of binary sequences was performed for length $N = 64$. The authors of these works presented tables with identical numbers of the obtained minimax sequences. In [6], the minimax sequences were found in the range of lengths $N = [49; 61]$, but the authors of this work presented only one example of the obtained sequences and did not mention their number. In [7], an exhaustive search for minimax sequences was performed in the range of lengths up to $N \leq 74$ and their number was reported.

Computational complexity of our global algorithm for finding binary sequences with given levels of the side lobes

$PSL = 2$	$PSL = 3$	$PSL = 4$	$PSL = 5$
$O(22.9 \times 1.42^N)$	$O(19.9 \times 1.6^N)$	$O(8.1 \times 1.69^N)$	$O(1.79^N)$

The following minimax sequences were constructed with the use of local search algorithms. In [8], binary sequences were found for lengths $N = 51, 69, 88$ with side lobe levels $PSL = 3, 4, 5$, respectively. In [5], the list of binary sequences was extended to length $N = 70$. In [9], binary sequences with side lobe levels $PSL = 4$ were found up to length $N = 82$, while those with side lobe levels $PSL = 5$ were found for lengths $N = [83, 105]$.

Let us describe a new global algorithm for searching for minimax binary sequences. The search space can be reduced by eliminating equivalent sequences. There are three transformations that preserving the maximum level of side lobe values: reversion $R(s_n) = s_{N-1-n}$, reversal of the sign to $N(s_n) = -s_n$, and an alternative change in sign to $S(s_n) = (-1)^n s_n$. These sequences form a class of equivalence.

The main difference of the new global search algorithm is that it does not take into account the asymmetry of the side lobes frames, and the values of the aperiodic ACF at shifts $\tau = N-2, N-3, \ldots, N/2$ are calculated after calculating them for shifts $\tau = 1, 2, \ldots, N/2$.

The results from calculating the experimental and theoretical computational complexity of the new algorithm for finding minimax sequences in the range of lengths $N = [10; 50]$ are shown in the figure. The number of calculations for aperiodic ACF frames is shown on the vertical axis in a logarithmic scale, while the length of a sequence is shown on the horizontal axis.

The computational complexity of the exhaustive search algorithm is determined as

$$O\left((N-1)2^N\right), \tag{8}$$

where 2^N is the number of possible binary sequences of length N, and $(N-1)$ is the number of frames of the aperiodic ACF side lobes. The experimental results from calculating the computational complexity are approximated according to the law

$$O(c \cdot b^N). \tag{9}$$

The computational complexity of the algorithm for finding binary sequences with side lobe levels $PSL = 2, 3, 4, 5$ is shown in the table.

For example, our result from a global search at length $N = 76$ is given below. The total number of nonequivalent optimum minimax sequences with $PSL = 4$ is equal to 18. Of these, the binary sequence in the form

1 1 –1 1 –1 –1 1 1 –1 –1 1 –1 –1 1 1 1 1 –1 –1 1 1
1 –1 1 1 –1 –1 –1 1 1 –1 1 1 –1 1 –1 1 –1 1 –1 1 –1
1 1 –1 1 1 1 1 –1 1 –1–1 –1 1 1 1 1 1 1 1 1 –1 –1 –1
1 1 –1 –1 –1 –1 1 –1 –1 –1 –1 1 1

has the maximum value of coefficient $MF = 7.113$.

At the same time, with a local search for the binary sequence with the maximum value of coefficient *MF* at this length, we found the binary sequence

1 1 1 1 1 1 1 –1 1 –1 1 1 1 1 1 –1 1 –1 1 –1 –1 –1 –1 1 1 1 1 –1 –1 –1 1 1 1 –1 –1 1 –1 1 –1 1 1 1 1 –1 –1 –1 1 –1 –1 –1 1 1 –1 –1 1 1 –1 1 1 1 –1 –1 1 –1 –1 1 –1 –1 1 –1 –1 1 1 1 –1 1 1

with the coefficient value $MF = 8.647$ and the maximum level of side lobes $PSL = 5$.

Detailed information on optimum binary sequences can be found at dedicated website [10].

CONCLUSIONS

The problem of constructing the ground states of a quantum system in the form of a sequence of pairwise interacting spins with length N was considered. It was shown that according to the Bernasconi model, this problem can be reduced to one of constructing binary sequences with the minimum energy levels of the side lobes of the aperiodic ACF. A new algorithm was proposed for global searches for the optimum minimax binary sequences. It has a computational efficiency comparable to that of other global algorithms. Synthesis of the optimum sequences for length $N = 76$ was presented as an example.

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